Path-integral approach to dynamics in a sparse random network

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We study the dynamics involved in a sparse random network model. We extend the standard mean-field approximation for the dynamics of a random network by employing the path-integral approach. The result indicates that the distribution of the variable is essentially identical to that obtained from globally coupled oscillators with random Gaussian interaction. We present the results of a numerical simulation of the Kuramoto transition in a random network, which is found to be consistent with this analysis.

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I. INTRODUCTION

Many systems in nature, such as food webs, metabolic systems, coauthorship of papers, the worldwide web, and so on, can be represented as complex networks [1–4]. Investigations of real networks have shown that these networks have topologies different from random networks. In particular, we have recognized that many networks have scale-free degree distribution, $P(k) \propto k^{-\gamma}$, where k is the degree of nodes.

The dynamics involved in complex networks has become an important aspect of the complex network studies in recent times. This problem includes, for example, the spreading of virus in the internet, synchronization of neurons in a brain, change of populations in a food web. Recently, Pastor-Satorras and Vespignani obtained unexpected results in this regard [5]. They studied the spread of viruses in complex networks and found that no threshold of infection rate exists for the susceptible-infected-susceptible model in the random scale-free network with $\gamma \leqslant 3$, if the size of the network N is infinite. Though real networks such as the internet are finite-size network, this result implies that a virus with a small infection rate can spread over the whole network. We had previously presented another remarkable example of the unusual dynamics involved in complex networks [6,7]. We studied the Kuramoto synchronization in a random network of oscillators and found that the critical coupling for synchronization becomes zero in scale-free network with $\gamma \leqslant 3$.

In these studies, the mean-field approximation plays an essential role. For the mean-field approximation, we consider a model in which a node i couples to another node j with a strength proportional to "mean coupling probability" $k_i k_j / k_{tot}$, where k_i and k_{tot} are the degree of node i and total number of edges, respectively. The dynamics in complex network is much simplified by this approximation, and we can obtain analytical results. However, this model differs from the original network model, in which each node couples to a finite number of nodes. It is remarkable that the mean-field approximation is in good agreement with the numerical simulation result of a random network model.

One of the objectives of this paper is to provide a sound explanation for the mean-field approximation. It is unclear why the mean-field approximation performs well in the random network model. The validity of the mean-field approximation, particularly with regard to the Kuramoto transition, is debatable. Moreno, Pacheco, and Vazquez-Prada carried out numerical simulations on the Kuramoto transition in the Barabási-Albert network [8,9]. They concluded that the critical coupling K_c is not 0 even if $N \to \infty$. Their conclusion seems to contradict the result of mean-field theory $K_c = 0$, though this discrepancy is possibly due to the difference of the order parameter used in these papers. Restrepo, Hunt, and Ott suggested that the the argument based on the largest eigenvalue of the network matrix is superior to that based on the mean-field theory [10]. They demonstrated that the mean-field approximation is valid for the Erdös-Rényi networks and random scale-free networks with $\gamma = 3$, while this approximation does not hold in the case of scale-free networks with $\gamma = 2$. However, they did not provide any explanation as to why the mean-field theory works well in some random network models. An appropriate explanation to this question is a matter of great interest and significance.

The second objective of this study is to extend the mean-field theory. Although the mean-field theory displays good qualitative coincidence with numerical simulations, it is impossible to examine the fluctuation of the variables by the mean-field approximation. Moereover, as noticed, we cannot apply the mean-field approximation in some network models. Therefore it is meaningful to make an approximation that covers a wider range of complex networks. In this paper, we demonstrate that the distribution of variables in the sparse random network model can be approximated by that obtained from a globally coupled network, in which the distribution of the interaction between the nodes is given by a Gaussian random number. This result indicates that the dynamics in random network can be approximated more precisely by appropriate methods such as dynamical mean-field theory [11].

In order to realize the above-mentioned objectives, we utilize the path-integral approach. The path integral, which was originally developed for application in quantum mechanics [12], has also been applied to random impurity problems [13,14], random spin glasses [15–17], neural networks [18,19], and oscillator systems [20]. One of the advantages

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of this approach is that the average over an ensemble of networks can be calculated easily. Limitations of the path-integral include an infinite number of integrals and obtaining a precise average over the ensemble, which is not usually possible. However, this method enables approximation of the distribution of the variables in a systematic manner. In particular, the mean-field approximation can be derived as the lowest order approximation of the path integral. The methods used by us are similar to that used by Theumann for the Hopfield network model [22].

The outline of this paper is as follows. In Sec. II, we present the general description of the dynamics of a network model based on the path-integral approach. We derive a formula that is general and can be applied to any network model in this section. In Sec. III we present two approximations of the path-integral formula, mean-field approximation, and perturbation. We also prove that the dynamics of a random network is essentially identical to that of a random Gaussian network. In Sec. IV, we apply the analysis to the Kuramoto transition in a random sparse network. We present the results of numerical simulation, which are consistent with that obtained from the analysis. To conclude, we discuss our obtained results.

II. PATH-INTEGRAL APPROACH TO THE DYNAMICS OF A NETWORK MODEL

In this section, we introduce the formalism to study the dynamics of a network model using the path-integral approach. We consider the following differential equations for the network model:

$$\dot{x}_i = f_i(x_i) + \sum_{j=1}^{N} a_{i,j} g(x_i, x_j) + \xi_i(t), \tag{1}$$

where $\xi_i(t)$ is a random force that satisfies $\langle \xi_i(t) = 0 \rangle$, $\langle \xi_i(t)\xi_j(t') \rangle = \delta_{i,j}\delta(t-t')\sigma^2$. We assume $x_i = x_{i,0}$ at $t=0$. In order to discuss the dynamics of this system, it is useful to introduce Matrin-Siggia-Rose (MSR) generating functional Z, which is defined as [13,14]

$$Z[\{l_{i,k}\},\{\bar{l}_{i,k}\}] = \left(\frac{1}{\pi}\right)^{NN_t} \left\langle \int \prod_{i=1}^{N} \prod_{k=0}^{N_t} dx_{i,k} d\bar{x}_{i,k} e^{-S} \times \exp(l_{i,k}x_{i,k} + \bar{l}_{i,k}\bar{x}_{i,k}) J \right\rangle, \tag{2}$$

where the action S is given by

$$S = \sum_{i,k} \left(\frac{\sigma^2 \Delta t}{2} \bar{x}_{i,k}^2 + i\bar{x}_{i,k} \left\{ x_{i,k} - x_{i,k-1} - \Delta t \left[f_i(x_{i,k-1}) + \sum_j a_{i,j} g(x_{i,k-1}, x_{j,k-1}) \right] \right\} \right), \tag{3}$$

and $\langle \cdots \rangle$ represents the average over the ensemble of networks. J is the functional Jacobian term,

$$J = \exp\left(-\frac{\Delta t}{2} \sum_{i,j,k} \frac{\partial [f_i(x_{i,k}) + a_{i,j} g(x_{i,k}, x_{j,k})]}{\partial x_{i,k}} \right). \tag{4}$$

Though this term is necessary for the renormalization $Z(0)=1$, it is a little cumbersome to treat it in a practical calculation, such as the mean-field approximation or a perturbation. Here we note that, as De Dominicis showed [14], the only effect of this Jacobian term is to subtract the nonretarded correlation function $\langle \bar{x}_{i,k} x_{j,k+k'} \rangle$, where $k' \geqslant 1$. In the following discussion, we omit this Jacobian term, remembering that we only consider the retarded correlation function. Maintaining $\Delta t N_t$ constant at the limit $\Delta t \to 0$, we obtain the MSR generating functional.

We consider the network described by

$$a_{i,j} = \begin{cases} 1 & \text{with probability } p_{i,j}, \\ 0 & \text{with probability } 1 - p_{i,j}. \end{cases} \tag{5}$$

We note that $p_{i,j}$ can be a function of variables such as i or j. For example, in the one-dimensional chain model, $p_{i,j}$ is 1 if $|i-j|=1$, otherwise it is 0. The average over all networks can be expressed as

$$\left\langle \exp\left[\sum_{i,k} i\Delta t \bar{x}_{i,k} \sum_j a_{i,j} g(x_{i,k-1}, x_{j,k-1}) \right] \right\rangle = \prod_{i,j} \left[p_{i,j} \exp\left\{ \sum_k i\Delta t \bar{x}_{i,k} g(x_{i,k-1}, x_{j,k-1}) \right\} + 1 - p_{i,j} \right], \tag{6}$$

and we obtain

$$\langle e^{-S} \rangle = \exp(-S_0) \prod_{i,j} \left[p_{i,j} \exp\left\{ \sum_k i\Delta t \bar{x}_{i,k} g(x_{i,k-1}, x_{j,k-1}) \right\} + 1 - p_{i,j} \right], \tag{7}$$

where

$$S_0 = \sum_{i,k} \frac{\sigma^2 \Delta t}{2} \bar{x}_{i,k}^2 + i\bar{x}_{i,k} \{ x_{i,k} - x_{i,k-1} - \Delta t f_i(x_{i,k-1}) \}. \tag{8}$$

The above-mentioned expression is a general one and can be applied to the dynamics of any network model. However, it is often impossible to calculate the precise value of $\langle e^{-S} \rangle$, particularly in the case of nonlinear dynamics. We need an approximation to obtain the value of $\langle e^{-S} \rangle$. In the next section, we approximate Eq. (6) by assuming $p_{i,k} \ll 1$ and $p_{i,j}p_{k,l} \ll p_{i,j}$ for any i, j, k, and l.

III. APPROXIMATION OF THE MSR GENERATING FUNCTIONAL IN A SPARSE RANDOM NETWORK MODEL

In this section, we develop an approximation for the MSR generating functional Z in a sparse random network. For this, we assume $p_{i,j} \ll 1$ and $p_{i,j}p_{k,l} \ll p_{i,j}$ for any i, j, k, and l. In the case of the Erdös-Rényi model, $p_{i,j}$ is independent of i and j; $p_{i,j} = \langle k \rangle / N$. Therefore this assumption is valid for a sparse Erdös-Rényi model, because $p_{i,j}p_{k,l} = \langle k \rangle^2 / N^2 \ll p_{i,j}$. In

the case of a random network with distribution $P(k)$, we construct the network as follows. First, we define the "degee" of node i as k_i', whose distribution concides with $P(k)$. Second, we connect the nodes i and j with probability $p_{i,j}=k_i'k_j'/\Sigma_i k_i'$. Using this procedure, we obtain the random network whose degree distribution is approximately given by $P(k)$. In this case, if the maximum degree of a node k_{max} is much smaller than N, $k_{max}\ll N$, the assumption is satisfied. On the other hand, this assumption is not satisfied in the Watts-Strogatz model, because $p_{i,i+1}\sim 1$.

Since $p_{i,j}\ll 1$, the approximate value of the logarithm of the right-hand side of Eq. (6) is expressed as follows:

$$\ln\left(\prod_{i,j}\left[p_{i,j}\exp\left\{\sum_k i\Delta t\bar{x}_{i,k}g(x_{i,k-1},x_{j,k-1})\right\}+1-p_{i,j}\right]\right)$$
$$\sim\sum_{i,j}-p_{i,j}+p_{i,j}\exp\left\{\sum_k i\Delta t\bar{x}_{i,k}g(x_{i,k-1},x_{j,k-1})\right\}$$
$$=\sum_{i,j}p_{i,j}\sum_{l=1}^{\infty}\frac{1}{l!}\left(\sum_k i\Delta t\bar{x}_{i,k}g(x_{i,k-1},x_{j,k-1})\right)^l. \quad (9)$$

Therefore, we obtain

$$\langle e^{-S}\rangle\sim\exp\left[\sum_{i,k}\left\{\frac{-\sigma^2\Delta t}{2}\bar{x}_{i,k}^2-i\bar{x}_{i,k}\{x_{i,k}-x_{i,k-1}-\Delta t f_i(x_{i,k-1})\}\right\}\right]\times\exp\left[\sum_{i,j}p_{i,j}\sum_{l=1}^{\infty}\frac{1}{l!}\left(\sum_k i\Delta t\bar{x}_{i,k}g(x_{i,k-1},x_{j,k-1})\right)^l\right]. \quad (10)$$

To calculate Z from this equation, we need an infinite number of integrals, and we cannot carry out this integration practically. However, this formula gives us much information about the averaged dynamics of networks. In the following subsections, we consider two simple approximation schemes, the mean-field approximation and perturbation.

A. Mean-field approximation and beyond

To begin with, we consider an approximation that ignores the $l\geqslant 2$ part of Eq. (10) and obtain

$$\langle e^{-S}\rangle\sim\exp\left(\sum_{i,k}\left\{-\frac{\sigma^2\Delta t}{2}\bar{x}_{i,k}^2-i\bar{x}_{i,k}\left[x_{i,k}-x_{i,k-1}-\Delta t(f_i(x_{i,k-1})+\sum_j p_{i,j}g(x_{i,k-1},x_{j,k-1}))\right]\right\}\right). \quad (11)$$

This result demonstrates that the MSR generating functional for Eq. (1) can be approximated as that for the system described by

$$\dot{x}_i=f_i(x_i)+\sum_j^N p_{i,j}g(x_i,x_j)+\xi_i(t). \quad (12)$$

This equation implies that the mean-field approximation neglects the contribution of the term $l\geqslant 2$ in Eq. (10). The mean-field approximation method is based on two assumptions. First, the higher-order term in $p_{i,j}$ in Eq. (9) is neglected, and, second, the higher-order term in Eq. (10) is neglected. The former assumption is valid if $p_{i,j}\ll 1$ for all values of i and j. However, neglecting the higher-order term is not always valid. In order to examine this argument, we study the effect of the term $l=2$. From the Stratonovich-Hubbard transformation, we obtain the following:

$$\exp\left[p_{i,j}\frac{1}{2}\left(\sum_k i\Delta t\bar{x}_{i,k}g(x_{i,k-1},x_{j,k-1})\right)^2\right]=\sqrt{\frac{1}{2\pi p_{i,j}}}\int dr_{i,j}\exp\left[-\frac{r_{i,j}^2}{2p_{i,j}}+i\left(r_{i,j}\sum_k\Delta t\bar{x}_{i,k}g(x_{i,k-1},x_{j,k-1})\right)\right]. \quad (13)$$

By comparing Eqs. (11)–(13), we observe that the MSR generating functional is identical to that of the system described by

$$\dot{x}_i=f_i(x_i)+\sum_j^N(p_{i,j}+r_{i,j})g(x_i,x_j)+\xi_i(t), \quad (14)$$

where $r_{i,j}$ is a random number and its distribution is given by a Gaussian distribution, with a mean value of 0 and a dispersion $\langle r_{i,j}^2\rangle=p_{i,j}$.

Sequential application of the Stratonovich-Hubbard transformation yields the contribution from the term $l=2^n$. For example, if we consider the term $l=4$, then because

$$\exp\left[p_{i,j}\frac{1}{4!}\left(\sum_k i\Delta t\bar{x}_{i,k}g(x_{i,k-1},x_{j,k-1})\right)^4\right]=\sqrt{\frac{3}{2\pi p_{i,j}}}\int dr'_{i,j}\exp\left[-\frac{3r_{i,j}'^2}{2p_{i,j}}+\frac{r'_{i,j}}{2}\left(\sum_k i\Delta t\bar{x}_{i,k}g(x_{i,k-1},x_{j,k-1})\right)^2\right], \quad (15)$$

the Gaussian fluctuation with dispersion $\sqrt{p_{i,j}/3}$ is added to $p_{i,j}$ in Eq. (13). Therefore, $\langle P\rangle$ is given by the solution of

$$\dot{x}_i=f_i(x_i)+\sum_j^N(p_{i,j}+r_{i,j})g(x_i,x_j)+\xi_i(t), \quad (16)$$

where the distribution of $r_{i,j}$ is given by a Gaussian with dispersion $\sigma_r=\sqrt{p_{i,j}+r'_{i,j}}$, and the distribution of $r'_{i,j}$ is also Gaussian with dispersion $\sigma'_r=\sqrt{p_{i,j}/3}$. By sequential application of this transformation, we can obtain the effect of the term $l=2^n$. However, as these correction terms are small for large l, the Gaussian random network is a good approximation of the random sparse network.

The estimation that utilizes the Stratonovich-Hubbard transformation is very effective in mapping the dynamics of the sparse network ensemble onto the dynamics of globally coupled networks. This method is very useful, especially in the case where the dynamics of Gaussian random networks is well known. However, the effectiveness of this transforma-

tion is limited, because it can only consider the terms $l=2^n$. This method does not elaborate on the effect of the terms $l=3,5,6,\ldots$. In the next subsection we realize that the correction in Z resulting from the term $l=2m+1$ is of the order $p_{i,j}^2$ in the random network.

B. Perturbation

As shown in the preceding section, though the mean-field approximation and the Stratonovich-Hubbard transformation are very effective methods, they only consider a limited number of terms. To examine the effect of other tems, we use the perturbation technique for a network model in this section. The perturbation gives a formal estimate of the value of $\langle e^{-S}\rangle$. Furthermore, we note that this method is highly effective because it allows us to estimate the accuracy of the approximation by an order of $p_{i,j}$. However, this method has two drawbacks. First, it is typically impossible to obtain the value of $\langle e^{-S}\rangle$ in nonlinear physics. It is often difficult to obtain the value of $\langle e^{-S}\rangle$ even if there is no interaction, and the perturbation can therefore only be applied to limited systems. Second, bifurcation or phase transition cannot be obtained without including the infinite order of perturbation in $p_{i,j}$. In general, the MSR generating functional Z becomes singular at the bifurcation point. However, the finite-order perturbation yields Z, which is a nonsingular function of $p_{i,j}$. Therefore, it is impossible to examine a phase transition by perturbation. However, perturbation often yields important information. In this section, we demonstrate that the correction from the odd l term is of the order $p_{i,j}^2$.

We begin with Eq. (10). On expanding $\exp(\Sigma p_{i,j}\cdots)$, we obtain

$$\langle e^{-S}\rangle = \exp\left[\sum_{i,k}\left\{-\frac{\sigma^2\Delta t}{2}\bar{x}_{i,k}^2 - i\bar{x}_{i,k}\{x_{i,k}-x_{i,k-1} - \Delta t f_i(x_{i,k-1})\}\right\}\right] \times \sum_{m=0}^{\infty}\frac{1}{m!}\left[\sum_{i,j}p_{i,j}\sum_{l=1}^{\infty}\frac{1}{l!}\left(\sum_k i\Delta t\bar{x}_{i,k}g(x_{i,k-1},x_{j,k-1})\right)^l\right]^m. \tag{17}$$

We first consider the correction from the term $l=3$. Since the effect of the term $l=2$ can be expressed as the fluctuation in $p_{i,j}$, we consider

$$e^{-S} \sim \exp\left(\sum_{i,k}\left\{-\frac{\sigma^2\Delta t}{2}\bar{x}_{i,k}^2 - i\bar{x}_{i,k}\left[x_{i,k}-x_{i,k-1}-\Delta t(f_i(x_{i,k-1}) + \sum_j p'_{i,j}g(x_{i,k-1},x_{j,k-1}))\right]\right\} + \sum_{i,j}\frac{p_{i,j}}{3!}\left(\sum_k i\Delta t\bar{x}_{i,k}g(x_{i,k-1},x_{j,k-1})\right)^3\right) \tag{18}$$

To account for the effect of the term $l=3$, we treat this term using perturbation. We expand the term $l=3$ as

$$\exp\left[\sum_{i,j}\frac{p_{i,j}}{3!}\left(\sum_k i\Delta t\bar{x}_{i,k}g(x_{i,k-1},x_{j,k})\right)^3\right] = 1+\sum_{i,j}\frac{p_{i,j}}{3!}\left(\sum_k i\Delta t\bar{x}_{i,k}g(x_{i,k-1},x_{j,k-1})\right)^3 + \frac{1}{2}\left\{\sum_{i,j}\frac{p_{i,j}}{3!}\left(\sum_k i\Delta t\bar{x}_{i,k}g(x_{i,k-1},x_{j,k-1})\right)^3\right\}^2+\cdots. \tag{19}$$

Here we assume $p_{i,j}p_{k,l}\ll p_{i,j}$ for any value of i, j, k, and l again.

Based on this assumption, $\langle e^{-S}\rangle$ can be approximated as

$$\langle e^{-S}\rangle = \left\{1+\sum_{i,j}\frac{p_{i,j}}{3!}\left(\sum_k i\Delta t\bar{x}_{i,k}g(x_{i,k-1},x_{j,k-1})\right)^3\right\}e^{-S_1}, \tag{20}$$

where

$$S_1 = \sum_{i,k}\left\{\frac{\sigma^2\Delta t}{2}\bar{x}_{i,k}^2 + i\bar{x}_{i,k}\left[x_{i,k}-x_{i,k-1}-\Delta t\left(f_i(x_{i,k-1}) + \sum_j p'_{i,j}g(x_{i,k-1},x_{j,k-1})\right)\right]\right\}. \tag{21}$$

To calculate the contributions to Z from these terms, it is convenient to define

$$P(\{\epsilon_{i,k}\}) = \left(\frac{1}{\pi}\right)^{NN_t/2}\int\prod_{i,k}d\bar{x}_{i,k}\exp\left(\sum_{i,k}\left\{-\frac{\sigma^2\Delta t}{2}\bar{x}_{i,k}^2 - i\bar{x}_{i,k}\left[x_{i,k}-x_{i,k-1}-\Delta t\left(f_i(x_{i,k-1}) + \sum_j p'_{i,j}g(x_{i,k-1},x_{j,k-1})-\epsilon_{i,k}\right)\right]\right\}\right). \tag{22}$$

Since

$$\int\prod_{i,k}d\bar{x}_{i,k}(-i\Delta t)^3\bar{x}_{i_1,k_1}\bar{x}_{i_1,k_2}\bar{x}_{i_1,k_3}e^{-S_1} = \frac{\partial^3}{\partial\epsilon_{i_1,k_1}\partial\epsilon_{i_1,k_2}\partial\epsilon_{i_1,k_3}}P(\{\epsilon_{i,k}\})\big|_{\epsilon_{i,k}\to 0}, \tag{23}$$

the MSR generating functional Z can be calculated from Eq. (20) if the differential of P is known. We first consider the differential of P in the case where k_1, k_2, and k_3 are distinct. On integrating Eq. (22), we obtain

$$P(\{\epsilon_{i,k}\}) = C\exp\left[-\sum_{i,k}\frac{1}{2\sigma^2\Delta t}\left\{x_{i,k}-x_{i,k-1}-\Delta t\left(f_i(x_{i,k-1}) + \sum_j p'_{i,j}g(x_{i,k-1},x_{j,k-1})-\epsilon_{i,k}\right)\right\}^2\right], \tag{24}$$

where $C=(1/\sigma^2\Delta t)^{NN_t/2}$. We note that the integration of $P(\{\epsilon_{i,k}\})$ over $x_{i,k}$ is $O(1)$, while $P(\{\epsilon_{i,k}\})$ is $O((\Delta t)^{3/2})$. On differentiating Eq. (24) with respect to ϵ_{i_1,k_1}, we obtain

$$\left.\frac{\partial P(\{\epsilon_{i,k}\})}{\partial \epsilon_{i_1,k_1}}\right|_{\epsilon=0} = \frac{C}{\sigma^2}\left\{x_{i_1,k_1} - x_{i_1,k_1-1} - \Delta t\left(f_{i_1}(x_{i_1,k_1-1}) + \sum_j p'_{i_1,j} g(x_{i_1,k_1-1}, x_{j,k_1-1})\right)\right\} e^{-S'_1}, \quad (25)$$

where

$$S'_1 = \sum_{i,k} \frac{1}{2\sigma^2 \Delta t}\left\{x_{i,k} - x_{i,k-1} - \Delta t\left(f_i(x_{i,k-1}) + \sum_j p'_{i,j} g(x_{i,k-1}, x_{j,k-1})\right)\right\}^2. \quad (26)$$

On differentiating Eq. (25) with respect to ϵ_{i_1,k_2} and ϵ_{i_1,k_3} we obtain $\partial^3 P / \partial \epsilon_{i_1,k_1} \partial \epsilon_{i_1,k_2} \partial \epsilon_{i_1,k_3}$. However, we temporarily consider the term $\partial P / \partial \epsilon_{i,k}$, because further differentiations with respect to ϵ_{i_2,k_2} and ϵ_{i_3,k_3} do not modify the following discussion.

In the limit $\Delta t \to 0$, $\exp(-S'_1)$ approaches to the δ function $\delta(x_{i,k} - x_{i,k-1} - \Delta t[f_i(x_{i,k-1}) + \Sigma_j p'_{i,j} g(x_{i,k-1}, x_{j,k-1})])$. Therefore, in the limit $\Delta t \to 0$, Eq. (25) always attains the value 0. However, $\Delta t N_t$ is maintained constant when the limit $\Delta t \to 0$ is taken. Therefore, the sum of the integrals of $\partial P / \partial \epsilon_{i,k}$ over $x_{i,k}$ may have a finite value at the limit $\Delta t \to 0$, if Eq. (25) has a magnitude $O(\Delta t)$. In order to obtain a more accurate estimate of Eq. (25), we express S'_1 as

$$\begin{aligned} S'_1 = \sum_{i,k} \frac{1}{2\sigma^2 \Delta t}\Bigg\{ & x_{i,k} - x_{i,k-1} - \Delta t\left(f_i(x_{i,k-1}) + \sum_j p'_{i,j} g(x_{i,k-1}, x_{j,k-1})\right)\Bigg\}^2 + \frac{1}{\sigma^2}\left(f_i(x_{i,k}) + \sum_j p'_{i,j} g(x_{i,k}, x_{j,k})\right)(x_{i,k+1} - x_{i,k}) + O(\Delta t) \\ & + (\text{the terms that do not include } x_{i,k}). \end{aligned} \quad (27)$$

We consider the integral $\int dx_{i,k} h(x_{i,k}) \partial P / \partial \epsilon_{i,k}$, where $h(x_{i,k})$ is an arbitrary nonsingular function. Since the value of $h(x_{i,k}) \times [x_{i,k} - x_{i,k-1} - \Delta t\{f_i(x_{i,k-1}) + p'_{i,j} g(x_{i,k-1}, x_{j,k-1})\}]$ is 0 at $x_{i,k} = x_{i,k-1} + \Delta t[f_i(x_{i,k-1}) + p'_{i,j} g(x_{i,k-1}, x_{j,k-1})]$, we introduce $y_{i,k} = x_{i,k} - x_{i,k-1} - \Delta t\{f_i(x_{i,k-1}) + p'_{i,j} g(x_{i,k-1}, x_{j,k-1})\}$, and expand using the Taylor expansion.

$$h(x_{i,k}) \exp\left\{\frac{1}{\sigma^2}\left(f_i(x_{i,k}) + \sum_j p'_{i,j} g(x_{i,k}, x_{j,k})\right)(x_{i,k+1} - x_{i,k}) + O(\Delta t) + \cdots\right\} = \sum_{m=1}^{\infty} h_m y_{i,k}^m. \quad (28)$$

Therefore, we obtain

$$\begin{aligned} \int dx_{i,k} h(x_{i,k}) \left.\frac{\partial P}{\partial \epsilon_{i,k}}\right|_{\epsilon=0} &= C \int dy_{i,k} \sum_m h_m y_{i,k}^m \exp(-y_{i,k}^2 / 2\sigma^2 \Delta t) \\ &= C \int dy'_{i,k} \sum_m (2\sigma^2 \Delta t)^{(m+1)/2} h_m y'^m_{i,k} \exp(-y'^2_{i,k}). \end{aligned} \quad (29)$$

Since the value of this integral becomes 0 if m is odd, the leading order of this integral is obtained from the term $m = 2$, and has a magnitude of $O(\Delta t^{3/2})$. Since the magnitude of this contribution is smaller than $O(\Delta t)$, we can neglect this term at the limit $\Delta t \to 0$.

Therefore, if k_1, k_2, and k_3 are unequal, then the value of the integral of the second term in Eq. (20) becomes 0. In the case where $k_1 = k_2 = k_3$, $\partial^3 P(\epsilon) / \partial \epsilon^3$ is negligible as $\Delta t \to 0$ from a similar argument. In general, the contribution from the term $\partial^m P(\epsilon) / \partial \epsilon^m$ is negligible for odd m, because it is expressed in the form $e^{-S'_1}\{x_{i,k} - x_{i,k-1} - \Delta t[f_i(x_{i,k-1}) + \Sigma_j p'_{i,j} g(x_{i,k-1}, x_{j,k-1})]\} \times (\text{nonsingular function})$.

Similarly, we can prove that $[\partial^m / (\partial \epsilon_{i_1,k_1} \cdots \partial \epsilon_{i_m,k_m})] P$ is nonzero if and only if each $\epsilon_{i,k}$ appears $2n$ times in the delimiter. From these results, we conclude that the contribution of the term $l = 3$ to Z is of the order $p_{i,j}^2$. In addition, we conclude that the contribution from the term $l = 2m + 1$ is of the order $p_{i,j}^2$. The contribution from the term $l = 2^m(2m' + 1)$ is also estimated by the Stratonovich-Hubbard transformation. In the case of the random network model, the correction due to these terms is small. For example, the contribution from the term $l = 6$ to the MSR generating functional term is estimated using

$$\begin{aligned} \exp\left[p_{i,j} \frac{1}{6!}\left(\sum_k i \Delta t \bar{x}_{i,k} g(x_{i,k-1}, x_{j,k-1})\right)^6\right] &= \sqrt{\frac{180}{\pi p_{i,j}}} \int dr_{i,j} \exp\left[-\frac{r_{i,j}^2}{p_{i,j}} + i r_{i,j}\left(\sum_k i \Delta t \bar{x}_{i,k} g(x_{i,k-1}, x_{j,k-1})\right)^3\right]. \end{aligned} \quad (30)$$

The dispersion of $r_{i,j}$ is $\sqrt{p_{i,j}/90}$ and the contribution of this term is much smaller compared to that from the Gaussian fluctuation obtained from the term $l = 2$.

From the discussion based on the Stratonovich-Hubbard transformation and perturbation, we demonstrated that the MSR generating functional for the dynamics of a random sparse network model is almost identical to that for the dynamics of a random Gaussian network. In the next section, we demonstrate that the above analysis is consistent with the result of a numerical simulation of the Kuramoto transition in a network model.

IV. EXAMPLE: THE KURAMOTO TRANSITION

In the previous section, we developed a general scheme to approximate the dynamics of the random sparse network and found its dynamics can be described by

$$\dot{x}_i = f_i(x_i) + \sum_j^N (p_{i,j} + r_{i,j}) g(x_i, x_j) + \xi_i(t) \tag{31}$$

when $p_{i,j} \ll 1$. In this case, the distribution of $r_{i,j}$ is provided by a Gaussian with dispersion $\sigma = \sqrt{p_{i,j}}$. In this section, we apply this approximation to the dynamics of oscillators in random networks.

We consider a random network of oscillators

$$d\theta_i/dt = \omega_i + K \sum_j a_{i,j} \sin(\theta_j - \theta_i) \tag{32}$$

where θ_i and ω_i represent the phase and velocity of the oscillator i, respectively. The value of $a_{i,j}$ is 1 if nodes i and j are connected, otherwise it is 0. We note that the random Gaussian matrix needs to be considered as symmetric. We consider the case where the distribution of ω_i is given by $g(\omega) = (1/\sqrt{2\pi\sigma_\omega}) \exp[(\omega - \omega_0)^2 / 2\sigma_\omega^2]$, and $a_{i,j}$ represents a random network with mean degree k_0. The above discussion suggests that the dynamics of this network can be approximated using the following equation:

$$d\theta_i/dt = \omega_i + \sum_j \left(\frac{k_0}{N} + r_{i,j} \right) K \sin(\theta_j - \theta_i), \tag{33}$$

where the distribution of $r_{i,j}$ is given by $P(r_{i,j}) = \sqrt{N/2\pi k_0} \exp[-N r_{i,j}^2 / (2k_0)]$. This model is similar to the dynamic glass model proposed by Daido [21]. However, the mean interaction between oscillators is positive in our model, while it is 0 in Daido's model.

It is difficult to calculate analytically the dynamics of this globally coupled model. In this section, we present the numerical results for the random sparse and random Gaussian networks. For this simulation, we set $N = 1000$, $2\sigma_\omega^2 = 1.0$, $\omega_0 = 0$, and $k_0 = 10$. The result obtained is averaged over 50 different networks.

First we examine the coupling dependence of the order parameter r. In our previous paper, we defined the order parameter r as $r = \Sigma_i k_i e^{i\theta_i} / \Sigma k_i$ for a random sparse network model. However, it is difficult to define such an order parameter for a random Gaussian model. In this paper, we therefore use $\langle e^{i\theta} \rangle$ as the order parameter for a random Gaussian network, and $r = \Sigma_i k_i e^{i\theta_i} / \Sigma k_i$ for a random sparse network model. Although these two order parameters are distinct, the difference between them is small, because the distribution of degree has a strong peak at $k = k_0$ for a random sparse network. The values of r are plotted in Fig. 1 for both the networks for the range $K = 0.02 - 0.20$. In both these models, the order parameter remains almost constant for K values less than 0.1. There is a rapid increase in r for K values greater than 0.1. The values of r coincide qualitatively for these two models. When $K = K_c$, a sharp transition, given by $r \propto \sqrt{K - K_c}$, is observed in the mean-field approximation.

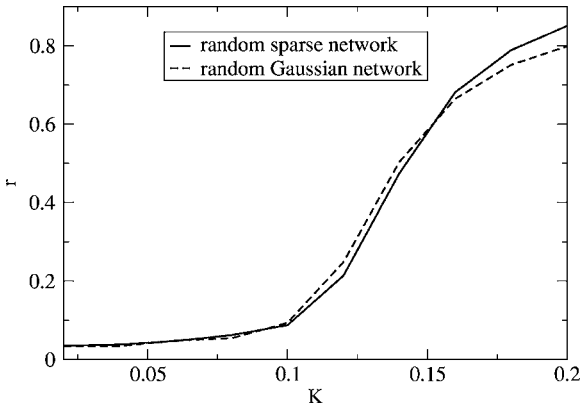

FIG. 1. The coupling dependence of the order parameter for random sparse and random Gaussian networks.

This sharp transition gets smeared out in a random sparse network. The Gaussian model approximates this smearing well.

The order parameters being identical is not unusual, because their obtained values were close to those obtained using the mean-field theory. We now explain the distribution of velocity $d\theta_i/dt$. In the mean-field approximation, $d\theta/dt$ has a δ-function-like peak at $d\theta/dt = 0$. However, if the coupling between the oscillators is random, the strong peak at $d\theta/dt$ will get smeared. In Fig. 2, the distribution of $d\theta/dt$ for sparse random and random Gaussian networks is plotted. At $K = 0.02$, there is an absence of synchronization and the distribution of $d\theta/dt$ is Gaussian-like. On the other hand, the oscillators are well synchronized and the distribution has a strong peak at $d\theta/dt = 0$ when $K = 0.16$. For the present study, we focus on the distribution at $K = 0.10$. This value of K is close to the critical point, and we suggest that the large fluctuation appears at this point. In the case of sparse networks, the peak at $d\theta/dt = 0$ is sharper at $k = 0.10$ than at $k = 0.02$. The same tendency is observed in the case of a Gaussian network. For example, we observe that $P(-0.1 < d\theta/dt < 0.1) = 0.137$ for a sparse random network. This value is close to $P(-0.1 < d\theta/dt < 0.1) = 0.131$ obtained from a random Gaussian network. This consistency in the observed value suggests that a random sparse network can be approximated by a Gaussian random network.

Finally, we present the distribution of the phase θ for both networks. The phase distribution in the (ω, θ) plane at $K = 0.16$ is shown in Fig. 3. Although the coupling strength is sufficiently large for synchronization, the phase does not entirely lie on a single line obtaind from the mean-field approximation method, $\theta = \arcsin(\omega / K k_0 r)$. In order to observe the dispersion around the mean-field line, we present the phase distribution of oscillators with $|\omega| \leq 0.05$ in Fig. 4. In this region, $|\arcsin(\omega / K k_0 r)|$ is less than 0.05 and the ω dependence of the phase distribution can be neglected. In both the models, the phase distribution lies in a wide range of θ. The dispersion σ for these two figures is $\sigma^2 = 0.105$ and 0.122 for the random sparse and the random Gaussian network, respectively. Since these two values coincide qualitatively, the random Gaussian network is a good approximation of the random sparse network.

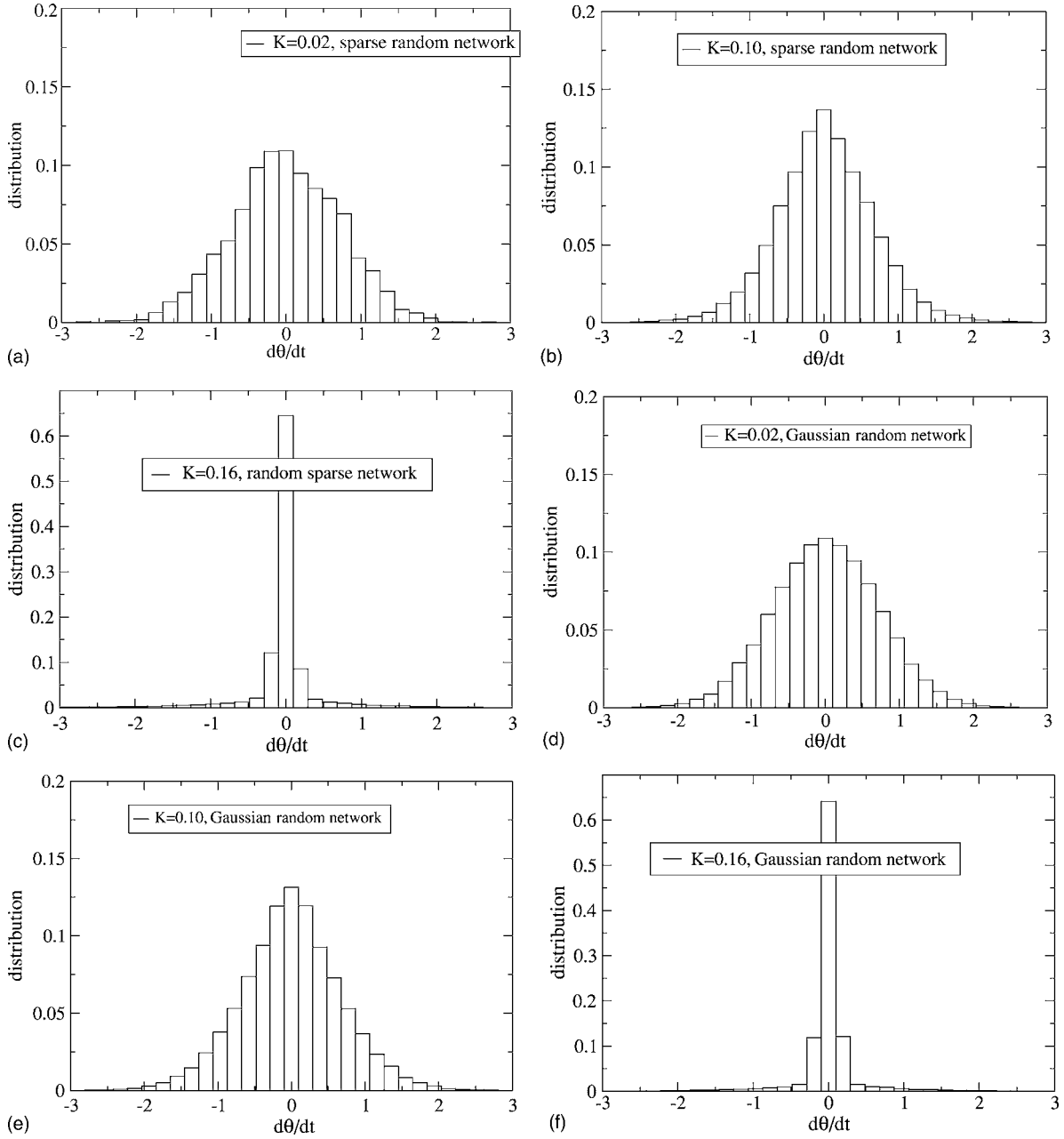

FIG. 2. The distribution of $d\theta/dt$ at $K=0.02$, 0.10, and 0.16 for a random sparse network (upper) and a random Gaussian network (lower).

V. CONCLUSION AND DISCUSSION

In this paper, we studied the dynamics of a random network model using the path-integral approach. We identified that the mean-field approximation is the lowest-order approximation of $p_{i,j}$ and $l=1$, as shown in Eq. (11). We also demonstrated that the contribution of the term $l=2^n$ can be described by the fluctuation of coupling in the globally coupled approximation method. The contribution of the odd l terms is difficult to estimate, though it is of the order $p_{i,j}^2$. We applied these general results to the Kuramoto transition, and observed a good agreement with numerical simulations.

The path-integral approach developed through this study is a general one and is applicable to dynamics of any random network. In particular, if the precise result for a randomly coupled model is known, a good approximation can be obtained for random sparse network models. There are several models, such as the replicator model [23,24], for which the exact results are known for a Gaussian random network. Our analysis proves that the dynamics of random sparse networks can be easily obtained for such models.

The analysis presented in this study is limited to the dynamics in a random network model. In the case of another network model, we need to include the higher-order terms to evaluate the MSR generating functional. It is usually difficult to carry out such a calculation. However, our result provide much information regarding the validity of the mean-field approximation. For example, the mean-field approximation

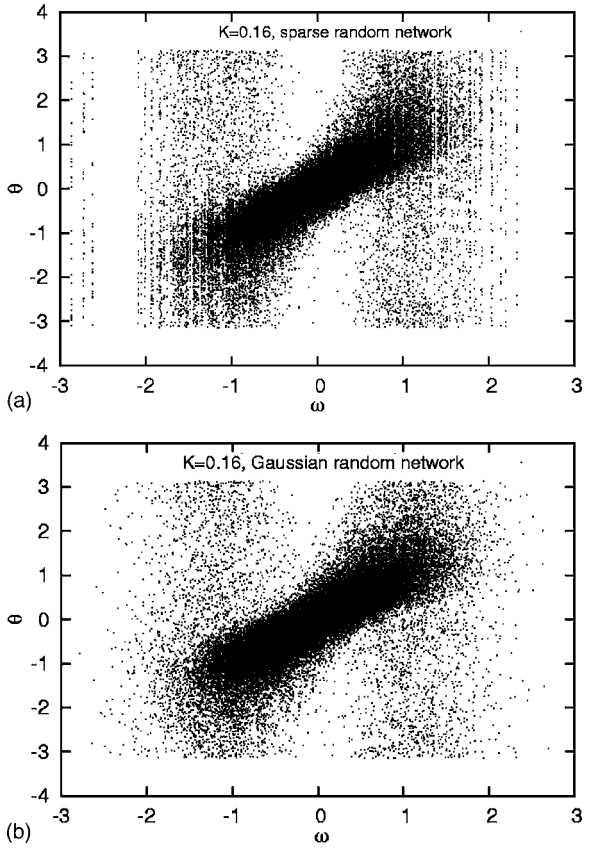

FIG. 3. Phase distribution in sparse and Gaussian random networks at $K=0.16$.

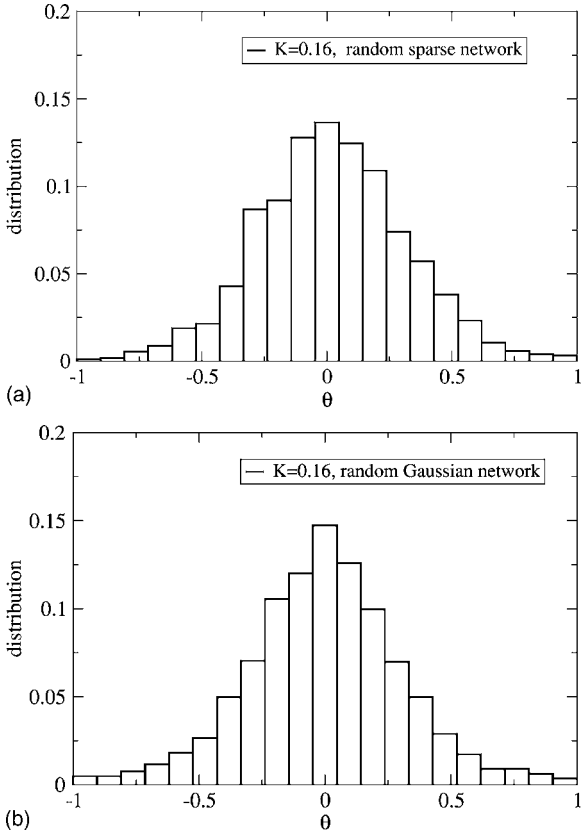

FIG. 4. Phase distribution in random sparse and random Gaussian networks at $K=0.16$.

is applicable if $p_{i,j}p_{k,l} \ll p_{i,j} \ll 1$. On the other hand, such an approximation is not applicable to the dynamics of a highly clustered network. In such a network, $p_{i,j}p_{j,k}p_{k,i} \sim O(p_{i,j}p_{j,k})$, $p_{i,j}p_{k,l} \ll p_{i,j}$ cannot be assumed and the contribution from the neglected terms needs to be calculated. It is usually believed that the dynamics of networks with high clustering coefficients cannot be approximated using the mean-field approximation method because of the high clustering coefficient. However, our analysis reveals that the validity of the mean-field approximation methods depends on the value of $p_{i,j}p_{k,l}$ and $p_{i,j}$. For example, the mean-field approximation method cannot be applied to the square-lattice model even if the clustering coefficient is zero, because the value of $p_{i,j}p_{k,l}$ can be as large as that of $p_{i,j}$.

We also discuss other studies conducted on the Kuramoto transition in random network models. Restrepo *et al.* examined the mean-field theory and studied the Kuramoto transition [10]. They concluded that synchronization occurs when K satisfies the relation $K > 2/\pi g(0)\lambda$, where λ is the largest eigenvalue of the network matrix $a_{i,j}$ and $g(0)$ is the density of the oscillators at $\omega=0$. They stated that the mean-field approximation, which was developed by us in previous papers, functions only when $r_i \propto k_i$, where r_i is the local field defined as $r_i=\langle \Sigma_j a_{i,j} e^{i(\theta_i-\theta_j)} \rangle_t$, where $\langle \cdots \rangle_t$ means the average over a long time interval. However, they did not explain the reason why this assumption is valid in some random network models, though they stated that there exists some relationship between the eigenvectors of $a_{i,j}$ and degree of each node. In this paper, we demonstrated that the mean-field theory is an approximation that considers only the term $l=1$ in the MSR generating functional. In this case, the mean-field approximation coincides with the discussion obtained from the largest eigenvalues, because the largest eigenvalue of the matrix $p_{i,j}=x_i x_j$ is $\Sigma_i x_i^2$ and its eigenvector v is given by $v=(x_1, x_2, \ldots, x_n)$. In the random network model, $p_{i,j}=k_i k_j/N\langle k \rangle$, where k_i and k_j are the degrees of the nodes i and j. Therefore, the largest eigenvalue of this matrix is $\langle k^2 \rangle / \langle k \rangle$, and the critical condition for synchronization in the mean-field approximation becomes identical to that in the discussion based on eigenvalues. In order to examine the applicability of the mean-field approximation, the term $l=2$ should be considered. In the case of a random matrix, the largest eigenvalue with a dispersion p is expressed by $2\sqrt{Np}=2\sqrt{\langle k \rangle}$ based on Wigner's semicircle law [24]. This result suggests that the mean-field approximation can be applied if $\sqrt{\langle k \rangle} \ll \langle k^2 \rangle / \langle k \rangle$. In order to examine this, we consider the matrix $M+G$, where M is the matrix obtained from the mean-field approximation and G the Gaussian random matrix, i.e., the distribution of each element of the matrix is Gaussian with dispersion $\sqrt{k}$. As observed earlier, the largest eigenvector v of matrix M satisfies the condition $Mv=\lambda v$, where $\lambda=\langle k^2 \rangle / \langle k \rangle$. On the other hand, $|Gv|$ is of the order $\sqrt{2k}|v|$, because all eigenvalues of G lie between $-\sqrt{2k}$ and $\sqrt{2k}$. Therefore, $|(M+G)v|$ equals approximately $\lambda|v|$ based on the assumption that $\lambda \gg \sqrt{k}$, and the direction of $(M+G)v$ is approximately identical to v. It should be noted that

all vectors u that are perpendicular to v, i.e., $(u,v)=0$, satisfy the condition $Mu=0$. This implies that $|(M+G)u| \leqslant \sqrt{2k}|u| \ll \lambda|u|$. Therefore, the largest eigenvalue and corresponding eigenvector of $M+G$ can be approximated as $\langle k^2\rangle/\langle k\rangle$ and $v=(k_1,\ldots,k_n)$, respectively. Therefore, the mean-field approximation is a suitable approximation if $\langle k\rangle$ is sufficiently large. In the case of a scale-free network, the spectrum density differs from that suggested by Wigner's law and the above-mentioned conclusion should be modified. However, this discussion suggests that the validity of the mean-field approximation is determined by the largest eigenvalues of the mean-field matrix M and random matrix G. If the largest eigenvalue of matrix G is as large as λ, the mean-field approximation is not valid. Based on the idea presented in this paper, the claim made by Restrepo *et al.* implies that the term $l=2$ must be included in order to discuss the critical behavior more accurately, especially in the case of a scale-free network with $\gamma=2$. Therefore, their work was not a denial of the mean-field theory, but an extension of it.

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[1] D. J. Watts and S. H. Strogatz, Nature (London) **393**, 440 (1998).
[2] R. Albert and A.-L. Barabási, Rev. Mod. Phys. **74**, 47 (2002).
[3] M. J. E. Newman, SIAM Rev. **45**, 167 (2003).
[4] S. N. Dorogovtsev and J. F. F. Mendes, *Evolution of Networks* (Oxford University Press, London, 2003).
[5] R. Pastor-Satorras and A. Vespignani, Phys. Rev. Lett. **86**, 3200 (2001).
[6] T. Ichinomiya, Phys. Rev. E **70**, 026116 (2004).
[7] T. Ichinomiya, Prog. Theor. Phys. **113**, 1 (2005).
[8] Y. Moreno and A. F. Pacheco, Europhys. Lett. **68**, 603 (2004).
[9] Y. Moreno, M. Vazquez-Prada, and A. F. Pacheco, Physica A **343**, 279 (2004).
[10] J. G. Restrepo, E. Ott, and B. R. Hunt, Phys. Rev. E **71**, 036151 (2005).
[11] J. A. Hertz, J. Phys. C **16**, 1219 (1983).
[12] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).
[13] P. C. Matrin, E. Siggia, and H. Rose, Phys. Rev. A **8**, 423 (1973).
[14] C. De Dominicis, Phys. Rev. B **18**, 4913 (1978).
[15] H. J. Sommers, Phys. Rev. Lett. **58**, 1268 (1987).
[16] H. Sompolinsky and A. Zippelius, Phys. Rev. B **25**, 6860 (1982).
[17] A. Crisanti and H. Sompolinsky, Phys. Rev. A **36**, 4922 (1987).
[18] R. Kree and A. Zippelius, Phys. Rev. A **36**, 4421 (1987).
[19] J. Balakrishnan, Eur. Phys. J. B **15**, 679 (2000).
[20] J. C. Stiller and G. Radons, Phys. Rev. E **58**, 1789 (1998).
[21] H. Daido, Phys. Rev. Lett. **68**, 1073 (1992).
[22] W. K. Theumann, Physica A **328**, 1 (2003).
[23] S. Diederich and M. Opper, Phys. Rev. A **39**, 4333 (1989).
[24] E. P. Wigner, Ann. Math. **62**, 548 (1955).